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Traffic Flow Models Considering an Internal Degree of Freedom

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We solve numerically the integrodifferential equation for the equilibrium case of Paveri–Fontana's Boltzmann-like traffic equation. Beside space and actual velocity, Paveri–Fontana used an additional phase space variable, the desired velocity, to distinguish between the various driver characters. We refine his kinetic equation by introducing a modified cross section in order to incorporate finite vehicle length. We then calculate from the equilibrium solution the mean-velocity–density relation and investigate its dependence on the imposed desired velocity distribution. A further modification is made by modeling the interaction as an imperfect showing-down process. We find that the velocity cumulants of the stationary homogeneous solution essentially only depend on the first two cumulants, but not on the exact shape of the imposed desired velocity distribution. The equilibrium solution can therefore be approximated by a bivariate Gaussian distribution which is in agreement with empirical velocity distributions. From the improved kinetic equation we then derive a macroscopic model by neglecting third and higher order cumulants. The equilibrium solution of the macroscopic model is compared with the cumulants of the kinetic equilibrium solution and shows good agreement, thus justifying the closure assumption.

KEY WORDS: Traffic dynamics; kinetic theory; fluid dynamics.

I. INTRODUCTION

Contributions to vehicular traffic theory are mainly put forward along to different lines. The macroscopic approach[1-3] comprises continuum models where, in analogy to hydrodynamics, one investigates the dynamics of

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averaged quantities like density, mean velocity and traffic flow. In the microscopic approach every vehicle is treated separately. Here, car-following models,[4,5] describing the dynamics of each individual car by differential-difference equations, and particle hopping models,[6-8] where space and time are discretized as a cellular-automata (CA), are quite popular. A third, and somewhat intermediate approach to traffic dynamics is the kinetic modeling,[9] where vehicles are treated similar to particles in gas kinetic theory. Recently, several articles[3,10] have been published dealing with the derivation of macroscopic traffic equations on the basis of a Boltzmann-like kinetic equation due to Paveri–Fontana[11] (PF). Paveri–Fontana modified Prigogine and Herman's[9] original approach to kinetic traffic modeling by the introduction of an additional phase space variable, the so-called "desired" velocity. In order to proceed to a macroscopic model one has to take moments of the kinetic equation. Usually the thus obtained hierarchy of moment equations is closed by approximating the local equilibrium solution of the kinetic equation with a normal distribution. The latter argument is based on empirical data and on the resemblance of the kinetic equation to the classical Boltzmann equation, but the equilibrium solution has not been investigated so far. Paveri–Fontana closed his article[11] with the remark "...that is does not seem possible to solve analytically... [the stationary and homogeneous problem], ...even for the simplest choices... [of the desired velocity distribution]. The development of computational schemes appears to be mandatory." Since the stationary homogeneous case plays as closure assumption a major role in the derivation of macroscopic models from a kinetic equation, we pick up Paveri–Fontana's thread in the present paper and investigate this problem numerically. Furthermore, we study modified versions of the PF collision operator which are introduced to overcome some of the main deficiencies of the original kinetic traffic equation. In the equilibrium case the improved kinetic equation yields fundamental diagrams and vehicle-velocity distributions comparable to traffic data. Based on this improved kinetic equation we derive a macroscopic model which yields realistic results over the whole density regime.

The paper is organized as follows: In Section II we briefly recapitulate Paveri–Fontana's kinetic traffic equation. We then investigate the time and space independent problem (Section III). Improvements on the kinetic equation are introduced and thoroughly investigated in Section IV for the stationary homogeneous case. The results are compared to traffic data and are related to other approaches. Based on the modified kinetic equation we derive a macroscopic model in Section V and compare the stationary homogeneous case with the corresponding results gained from the improved kinetic equation.

II. PAVERI–FONTANA'S EQUATION

Paveri–Fonata[11] proposed the following Boltzmann-like traffic equation:

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)g(x,v,w,t)+\frac{\partial}{\partial v}\left(\frac{w-v}{T}\,g(x,v,w,t)\right)$$
$$=f(x,v,t)\int_v^{+\infty}dv'(1-P)(v'-v)\,g(x,v',w,t)$$
$$-g(x,v,w,t)\int_0^v dv'(1-P)(v-v')\,f(x,v',t) \qquad (2.1)$$

Here, $g(x, v, w, t)$ denotes the one-vehicle distribution function for vehicles with desired speed w and actual speed v on the phase space spanned by x, v, w, t, with $g(x, v, w, t)\,dx\,dv\,dw$ being the number of vehicles at time t, in position dx around x and actual speed dv around v with desired speed dw around w. The one vehicle speed distribution function $f(x, v, t)$ is defined as:

$$f(x,v,t)=\int_0^{+\infty}dw\,g(x,v,w,t) \qquad (2.2)$$

The road is assumed to be a one dimensional, unidirectional lane, but passing is allowed. This can be conceived as a multi-lane road where an average over the different lanes has been taken. The term on the RHS of Eq. (2.1) represents the interactions due to "collisions," i.e., the slowing down process, where the following assumptions have been made:

(i) The slowing down process has a probability $(1-P)$, where P denotes the probability of passing, $0 \leqslant P \leqslant 1$. If the fast car passes the slow one, its velocity is not affected.

(ii) The velocity of the slow car is unaffected by the interaction or by the fact of being passed.

(iii) Cars are regarded as point-like objects, so the vehicle length can be neglected.

(iv) When two vehicles interact the fast car instantaneously adopts the velocity of its leading slower car, there is no braking time.

(v) Only two-vehicle-interactions are to be considered, multi-vehicle interactions are excluded.

(vi) One assumes "vehicular chaos," i.e., vehicles are not correlated,

$$g_2(x, v, w, x', v', w', t) \simeq g(x, v, w, t)\, g(x', v', w', t) \tag{2.3}$$

where g_2 denotes the two-vehicle distribution function.

The first part of the collision integral of Eq. (2.1) describes the gain of the phase space element, i.e., vehicles with velocity $v' \geqslant v$ collide with vehicles with velocity v, while the second term describes the loss of the phase space element, i.e., vehicles with velocity v collide with vehicles with even slower velocity v'. Furthermore, it is assumed that no driver changes his desired speed, i.e.,

$$\frac{dw}{dt} = 0 \tag{2.4}$$

and that the acceleration of each car is modeled by

$$\frac{dv}{dt} = \frac{w - v}{T} \tag{2.5}$$

i.e., the drivers approach their desired speed exponentially in time, with time constant T. (T might be a function of c, v, see for example ref. 12). Since no driver can achieve a velocity v greater than his desired velocity w, $g(x, v, w, t)$ is restricted to $g(x, v, w, t) \geqslant 0$ for $0 < v < w$. Each driver determines the acceleration of the vehicle by its individual desired speed, so Eq. (2.4) can be understood as an acceleration due to an internal force. In this sense we call the phase space variable w an additional (internal) degree of freedom, but now a desired velocity distribution has to be given. Since it seems impossible to measure the desired velocity distribution, one has to make appropriate assumption, for example a Gaussian distribution. On the other hand, it is obvious that imposed speed limits will have a major influence on at least the first two cumulants of the desired velocity distribution. Notice also that owing to the form of the interaction term, no vehicle can achieve a velocity less than the smallest desired velocity. This is one of the short-comes of Paveri–Fontana's equation. Another problem is, that with assuming "vehicular chaos" we neglect correlations between the vehicles, and therefore the kinetic equation is only valid for dilute traffic. Both of these restrictions will lead to modified versions of the traffic Eq. (2.1).

The vehicular concentration $c(x, t)$, the average velocity $\bar{v}(x, t)$, the average desired velocity $\bar{w}(x, t)$ and the flow $q(x, t)$ are defined as

$$c(x, t) = \int_0^{+\infty} dw \int_0^{+\infty} dv\, g(x, v, w, t) \tag{2.6}$$

$$\bar{v}(x, t)\, c(x, t) = \int_0^{+\infty} dw \int_0^{+\infty} dv\, vg(x, v, w, t) \tag{2.7}$$

$$=: q(x, t) \tag{2.8}$$

$$\bar{w}(x, t)\, c(x, t) = \int_0^{+\infty} dw \int_0^{+\infty} dv\, wg(x, v, w, t) \tag{2.9}$$

The probability of passing is usually chosen to be density dependent, and in the following we will use Prigogine and Herman's suggestion,[9] i.e., $P(c) = 1 - c/\hat{c}$ ($\hat{c}$ denotes the maximal density). Additional velocity and variance dependence can certainly be conceived (see ref. 12).

Regarding Paveri–Fontana's traffic equation as a nonlinear initial-value problem the existence and uniqueness has been proven in ref. 13. Based on this proof an iterative method was given by Semenzato[14] to find as solution of the time dependent problem as the limit of two sequences of solutions of linear equations. Therefore a unique solution exists also for the time- and position independent problem and we first turn our attention to this case. In order to solve this equation numerically we follow a direction already pointed out by Paveri–Fontana. It relies on the assumption that the density c and the normalized desired distribution function $H(w)$ are known. Another possible approach is based on a known current q and a known desired flux $\phi(w) = \int_0^{+\infty} dv\, vg(v, w)$.

III. THE STATIONARY HOMOGENEOUS CASE

Let us rewrite the stationary homogeneous version of Eq. (2.1) in a normalized form and therefore introduce functions H, G defined by

$$H(w)\, c := \int dv\, g(v, w), \qquad G(v \mid w) := \frac{g(v, w)}{H(w)\, c} \tag{3.1}$$

With Eq. (2.6), it immediately follows that

$$\int dw\, H(w) = 1, \qquad \int_0^w dv\, G(v \mid w) = 1 \tag{3.2}$$

$H(w)$ is the normalized desired velocity distribution and $G(v \mid w)$ is the conditional distribution for vehicles with desired velocity w. The normalized velocity distribution $F(v)$ is then given by

$$F(v) = \int_v^\infty dw\, G(v \mid w)\, H(w) \qquad \text{with} \quad \int_0^\infty dv\, F(v) = 1 \tag{3.3}$$

In terms of the normalized functions H, G the stationary homogeneous equations now reads

$$\frac{\partial}{\partial v}\left[(w-v)\, G(v \mid w)\right] = \alpha \int du\, G(v, u)\, H(u) \int_v^{+\infty} dv'(v'-v)\, G(v' \mid w)$$
$$-\alpha G(v \mid w) \int_0^v dv'(v-v') \int du\, G(v', u)\, H(u) \tag{3.4}$$

where $\alpha = cT(1-P)$. In Eq. (3.4) $G(v \mid w)$ is the unknown, whereas α and H are given with $H(w) \geqslant 0$ for $w > 0$ and $H(w) \equiv 0$ otherwise. The nonlinear integro-differential equation (3.4) has to be solved for $0 < v < w$ and subject to the requirement $G(v \mid w) \geqslant 0$.

In the collision-less case ($\alpha = 0$), it is quite clear that

$$G(v \mid w) = \delta(v - w) \tag{3.5}$$

is the wanted solution.

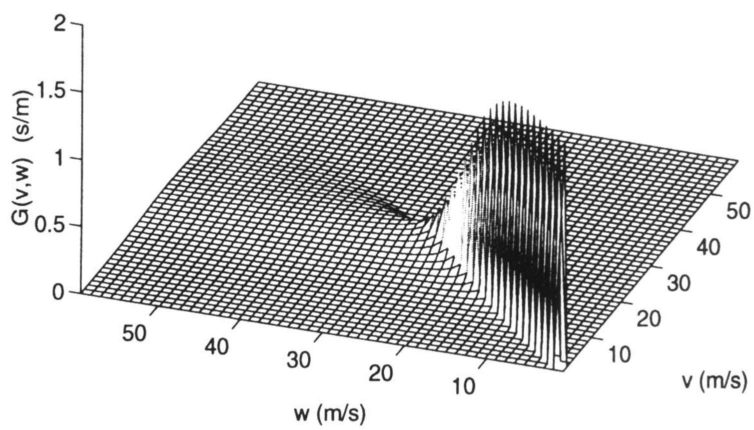


Fig. 1. Conditional velocity distribution $G(v \mid w)$ for vehicle with desired velocity w at $c = 0.3\hat{c}$.

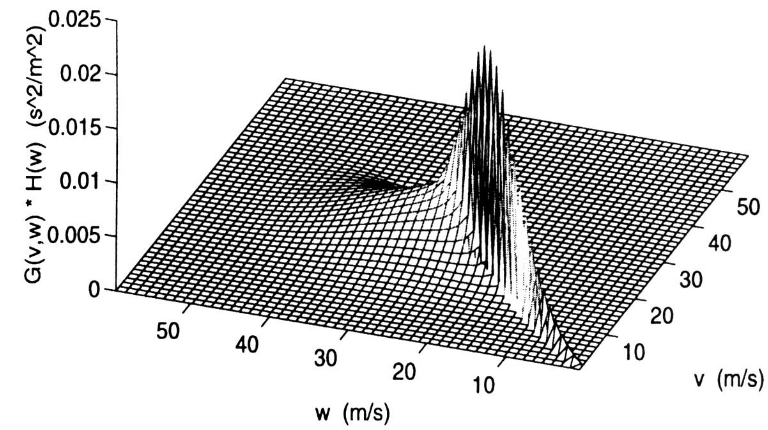


Fig. 2. Weighted distribution $G(v \mid w)\, H(w)$ for $c = 0.3\hat{c}$. $H(w)$ is a Gaussian with mean value $\bar{w} = 30$ m/s and variance $\Theta_{ww} = 86.84\ \mathrm{m}^2/\mathrm{s}^2$.

In order to investigate Eq. (3.4) numerically, we regard the integro-differential equation as a balance equation in the (v, w)-velocity space. We discretize Eq. (3.4) such that the vehicle number is conserved automatically and optimize an initial distribution $G_0(v, w)$ by minimizing the transition rates between the "velocity cells" calculated from Eq. (3.4), subject to $G(v \mid w) \geqslant 0$.

As a first set up we take $P(c) = 1 - c/\hat{c}$, where $\hat{c}$ is the maximal density and $T = 10s$. As desired velocity distribution $H(w)$ we choose a Gaussian with $\bar{w} = 30$ m/s and $\Theta_{ww} = 86.84\ \mathrm{m}^2/\mathrm{s}^2$ and $H(w) \equiv 0$ for $w \notin [0\ \mathrm{m/s}, 60\ \mathrm{m/s}]$. In Figs. 1, 2, 3, and 4 the normalized distribution $G(v \mid w)$ and the weighted distribution $G(v \mid w)\, H(w)$ are shown for $c = 0.3\hat{c}$ and $c = 0.6\hat{c}$, respectively. As already mentioned, the PF-equation only holds for dilute traffic, so the case $c = 0.6\hat{c}$ just serves to state the qualitative changes when proceeding to higher densities. One finds that the distribution G separates into two parts, one part is the "free" vehicles depicted by the delta-peak-like function on the diagonal, whereas the more spread out distribution in the high velocity regime represents the interacting cars.[2] This is intuitively quite clear, since the vehicles with small desired velocity can only collide with vehicles with even smaller actual velocity, therefore their interaction frequency is very low and they are thus able to drive almost with their desired velocity (confer the collision-less case Eq. (3.5)). On the other hand, for vehicles with a high desired velocity the interaction frequency is

[2] This reminds us to kinetic traffic equations, where free and queuing vehicles are treated separately from the very beginning, see for example ref. 15.

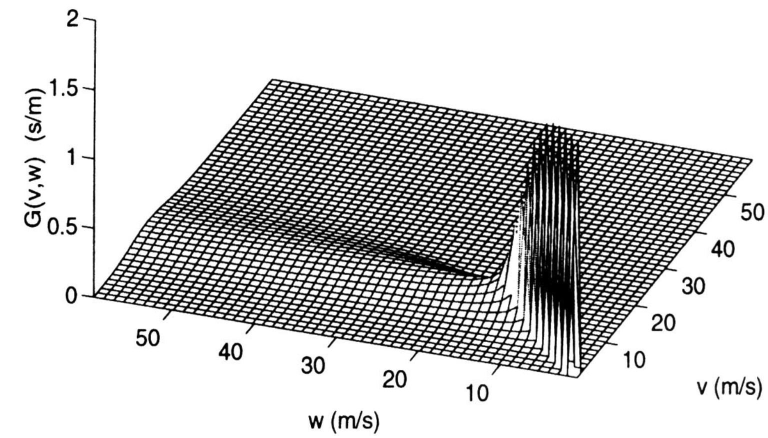


Fig. 3. Conditional velocity distribution $G(v \mid w)$ for vehicle with desired velocity w at $c = 0.6\hat{c}$.

higher and these vehicles are "scattered away" from their desired velocity to lower velocities. CA-models[6–8] show that the vehicles accumulate as free and congested traffic, but these different states are separated in space as stop-and-go waves. From a mathematical point of view, we here investigate the homogeneous case of a kinetic equation, so on a coarse grained scale the separation takes also place locally (remember that we are

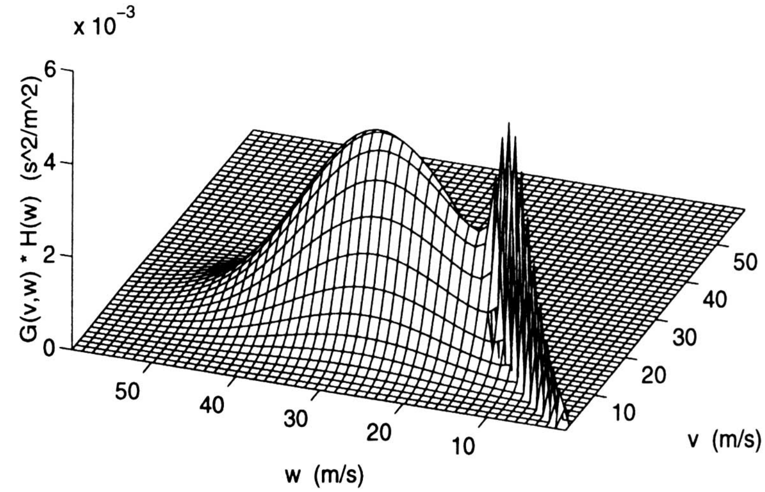


Fig. 4. Weighted distribution $G(v \mid w)\, H(w)$ for $c = 0.6\hat{c}$. $H(w)$ is a Gaussian with mean value $\bar{w} = 30$ m/s and variance $\Theta_{ww} = 86.84\ \mathrm{m}^2/\mathrm{s}^2$.

dealing with a quasi-one-dimensional model where an average over the lanes has already been taken.). As we expect, the relative number of interacting cars (free cars) increases (decreases) at higher densities. Since the PF-equations is only valid for dilute traffic we can not hope to get meaningful results when calculating the means velocity as a function of the density. We just remark that $\bar{v}(c)$ is a strictly monotonously decreasing function, but that the mean actual velocity does not go to zero for $c \to \hat{c}$.

IV. MODIFIED KINETIC TRAFFIC EQUATIONS

A. Enskog-like Model

In order to find a model yielding more realistic results over the whole density regime we now follow an argument used by Enskog[16, 17] when he introduced his theory of dense gases.[3] At medium and high densities the vehicles cannot be regarded as point-like objects any more, but have a spatial extension l (average vehicle length). This leads to two different modifications in the Boltzmann-like equation:

(i) The common position x of the two interacting vehicles in the collision integral [RHS(2.1)] should be replaced by the actual positions of the centers of the two vehicles and the "cross section" should be taken at the actual position of the collision.

(ii) Since the covolume of the cars is now comparable to the total volume of the system, the volume where the center of any vehicle can lie is reduced and therefore the collision frequency is enhanced.

Since we study only the homogeneous case we just have to consider the second effect. The effective volume is reduced by a factor $(1 - c/\hat{c})$, hence the scattering frequency is increased by a factor $\chi = (1 - c/\hat{c})^{-1}$. We then get an Enskog-like stationary homogeneous equation by replacing $(1 - P(c))$ in Eq. (3.4) through the modified cross section[4] $\sigma(c) = \chi(c)(1 - P(c))$. We now investigate the behavior of the modified equilibrium equation and its solution with respect to the width of the desired distribution function. Again we have chosen $P(c) = 1 - c/\hat{c}$. As desired velocity distribution $H(w)$ we use three Gaussian with equal mean values but different variances ($\Theta^{-}_{ww} = 86.84\ \mathrm{m^2/s^2}$,

[3] Enskog's considerations are heuristic, for a more thoroughly based model, see ref. 18. Nelson[19] proposed a correlation model for a kinetic traffic equation.

[4] Through this modified cross section σ, we replace the "vehicular chaos" assumption in Section II through the Ansatz $g_2 = \chi g_1 g_1$. Since the two-vehicle correlation function $\mathscr{C}_2$ is defined by $g_2 = g_1 g_1 + \mathscr{C}_2$, the introduction of the modified cross section is equivalent to the assumption $\mathscr{C}_2 = (\chi - 1)\, g_1 g_1 = g_1 g_1/(\hat{c}/c - 1)$.

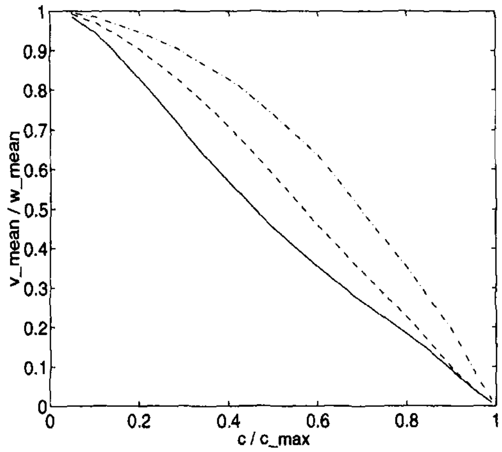


Fig. 5. Fundamental diagram for $\Theta_{ww}^{-} = 86.84\ \mathrm{m^2/s^2}$, $\Theta_{ww}^{--} = \frac{1}{2}\Theta_{ww}^{-}$ and $\Theta_{ww}^{-\cdot} = \frac{1}{4}\Theta_{ww}^{-}$.

$\Theta_{ww}^{--} = \frac{1}{2}\Theta_{ww}^{-}$, $\Theta_{ww}^{-\cdot} = \frac{1}{4}\Theta_{ww}^{-}$) and $H(w) \equiv 0$ for $w \notin [0\ \mathrm{m/s}, 60\ \mathrm{m/s}]$. Figure 5 shows the mean velocity-density relation and Fig. 6 shows the variance of the actual velocity as a function of the density. Both graphs go to zero for $c \to \hat{c}$, so at least the minimal requirement is satisfied. Next we find that for broader desired velocity distributions the fundamental diagram declines faster. Going from a low to a high desired velocity variance the mean velocity changes from a concave to a convex function of the density.

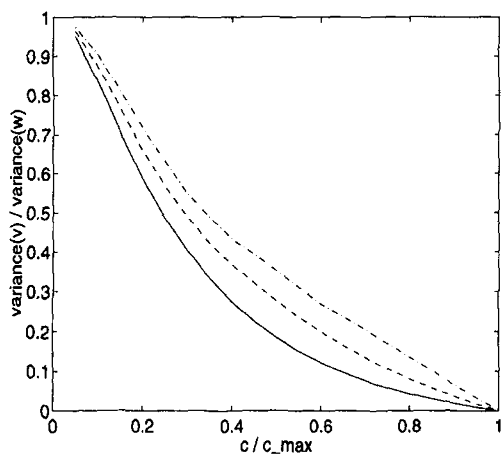


Fig. 6. Velocity variance as a function of density for $\Theta_{ww}^{-} = 86.84\ \mathrm{m^2/s^2}$, $\Theta_{ww}^{--} = \frac{1}{2}\Theta_{ww}^{-}$ and $\Theta_{ww}^{-\cdot} = \frac{1}{4}\Theta_{ww}^{-}$.

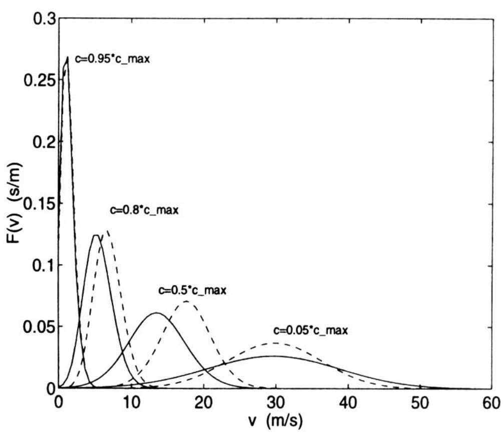


Fig. 7. Velocity distribution $F(v)$ at different densities: $\Theta_{ww}^{-} = 86.84\ \mathrm{m}^2/\mathrm{s}^2$, $\Theta_{ww}^{--} = \frac{1}{2}\Theta_{ww}^{-}$. (normal desired velocity distribution)

Figure 7 depicts the velocity distribution $F(v)$ for different densities. The obtained Gaussian shaped distributions are in agreement with empirical data.[3, 20, 21] In Fig. 8 empirical velocity distributions are compared to normal velocity distributions with the same mean value and variance. The critical reader who objects that this Gaussian shape might just be a result of the imposed normal desired distribution $H(w)$ is referred to the next section.

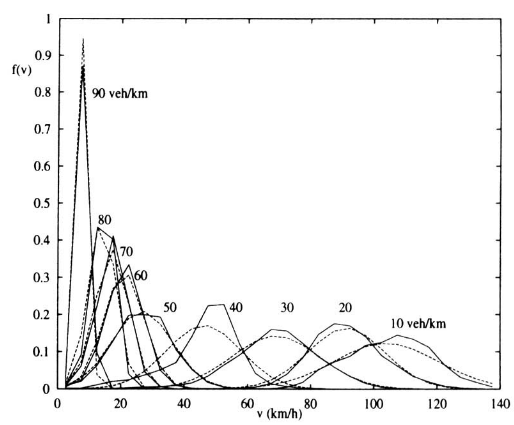


Fig. 8. Comparison of empirical velocity distributions at different densities (—) with Gaussian velocity distributions with the same mean value and variance (— —).[23]

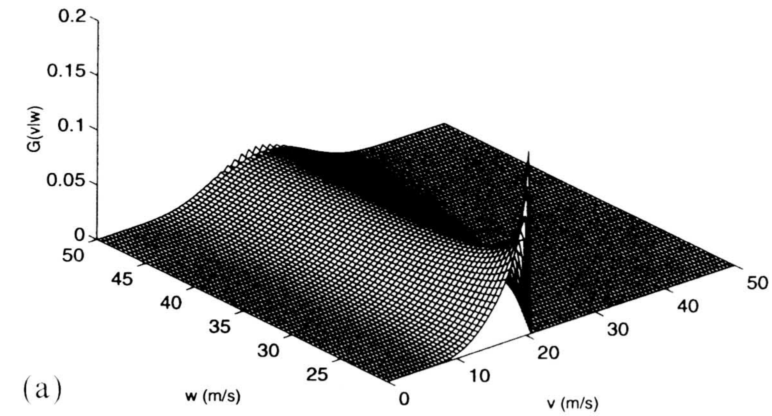
B. Imperfect Velocity Adaptation

In the idealized slowing down process modeled by the collision integral of Eq. (2.1), the fast vehicle instantaneously adopts the velocity of its leading vehicle. A short glance at Eq. (2.1) immediately shows that no vehicle can reach a velocity slower than the smallest desired velocity. In the former numerical experiment $H(w)$ was small but non-vanishing for small w, though it seems unrealistic to have drivers with such small and even zero desired velocities. Drivers, who are forced to slow down often over-react, so a more realistic interaction models should allow that the fast car also slows down to velocities smaller than the velocity of its leading vehicle. Heibing[3] for example, assumes that fast vehicles are scattered to velocities smaller than the leading-car velocity with exponentially decreasing probability and he calls this "imperfect driving." A similar effect has been introduced in the CA-model by Nagel and Schreckenberg[6] through a random deceleration step. Here, we allow fast vehicles to slow down to velocities $v \in [\beta v', v']$ with uniform probability (v' is the velocity of the leading vehicle and $0 < \beta < 1$ constant).[5] This simpler model possesses already the important features and suffices our purposes. The modified collision integral [RHS(3.4)] takes now the following form:

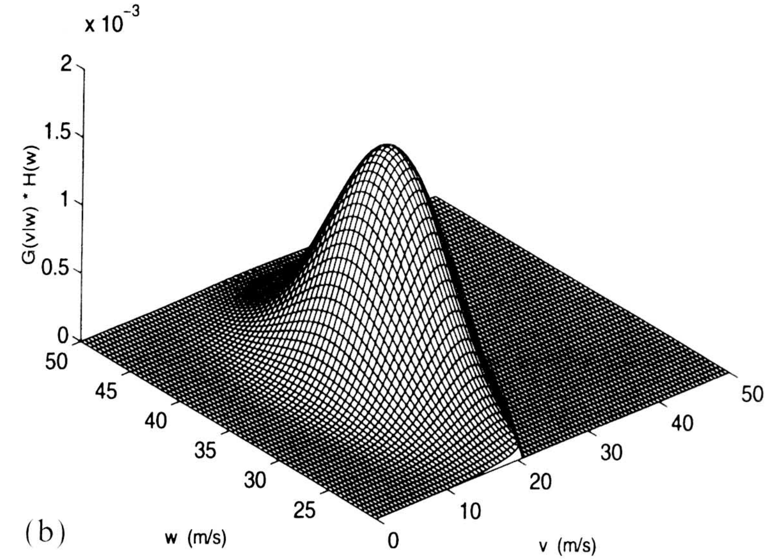
$$\begin{aligned}
\mathscr{I}_{coll} = {} & \alpha' \iint_{0 \leqslant v_1 \leqslant v_2} dv_1\, dv_2 (v_2 - v_1) \\
& \times \int du\, G(v_1 \mid u)\, H(u)\, G(v_2 \mid w) \frac{1}{v_1(1-\beta)} \hat{\chi}_{[\beta v_1, v_1]}(v) \\
& - \alpha' G(v \mid w) \int_0^v dv_1 \int_{v_2 \leqslant v_1} dv_2 (v - v_1) \\
& \times \int du\, G(v_1 \mid u)\, H(u) \frac{1}{v_1(1-\beta)} \hat{\chi}_{[\beta v_1, v_2]}(v_2) \\
= {} & \alpha' \int_v^{v/\beta} dv_1 \int du\, G(v_1 \mid u)\, H(u) \int_{v_1}^{\infty} dv_2 (v_2 - v_1)\, G(v_2 \mid w) \frac{1}{v_1(1-\beta)} \\
& - \alpha' G(v \mid w) \int_0^v dv_1 (v - v_1) \int du\, G(v_1 \mid u)\, H(u)
\end{aligned} \tag{4.1}$$

where $\alpha' = cT\sigma = cT\chi(c)(1 - P(c))$ and $\hat{\chi}_{[a,b]}$ is the characteristic function of the interval $[a, b]$. $1/[v_1(1-\beta)]$ is a normalization factor.

[5] Wegener and Klar[22] used a similar interaction to model the slowing down process in their kinetic traffic equation.



(a)



(b)

Fig. 9. (a) $G(v \mid w)$ normalized velocity distribution for vehicle with desired velocity w at $c = 0.4\hat{c}$ and $\beta = 0.8$. $H(w)$ is a Gaussian with $H(w) \equiv 0$ for $w \notin [20\ \text{m/s}, 50\ \text{m/s}]$, $\bar{w} = 35$ m/s and $\Theta_{ww} = 25.8\ \text{m}^2/\text{s}^2$. (b) Weighted distribution $G(v \mid w)\, H(w)$ for $c = 0.4\hat{c}$ and $\beta = 0.8$.

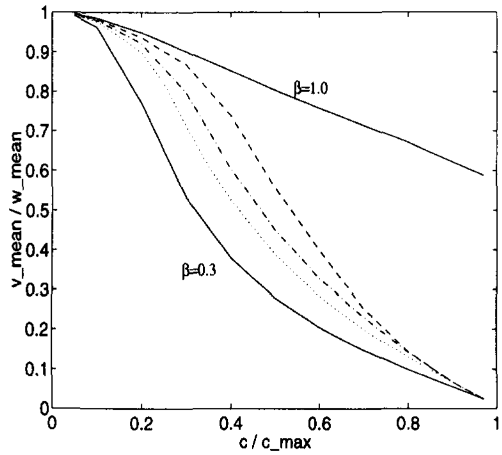


Fig. 10. Mean-velocity-density relation for different values of the parameter β. $\beta = 1.0$ upper solid line, $\beta = 0.9$ (— —), $\beta = 0.8$ (—·), $\beta = 0.7$ (···), $\beta = 0.3$ lower solid line.

In Figs. 9(a) and (b) we plot the normalized velocity distribution $G(v \mid w)$ for vehicles with desired velocity w and the weighted distribution function $G(v \mid w)\, H(w)$, respectively, at $c = 0.4\hat{c}$ and with $\beta = 0.8$. $H(w)$ is a Gaussian with $H(w) \cong 0$ for $w \notin [20\text{ m/s}, 50\text{ m/s}]$, $\bar{w} = 35$ m/s and $\Theta_{ww} = 25.8\text{ m}^2/\text{s}^2$. A rough comparison with Figs. 1–4 shows that the clear separation between free and interacting vehicles has almost disappeared. Besides, the weighted distribution $G(v \mid w)\, H(w)$ seems to be well approximated by

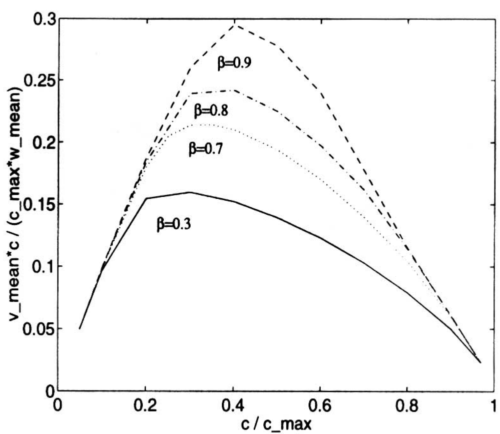


Fig. 11. Flux $c\bar{v}$ as a function of density: $\beta = 0.9$ (— —), $\beta = 08$ (—·), $\beta = 0.7$ (···), $\beta = 0.3$ (—).

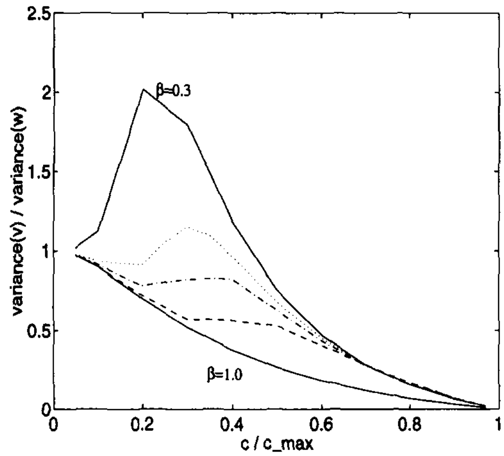


Fig. 12. Velocity-variance as a function of density: $\beta = 1.0$ lower solid line, $\beta = 0.9$ (— —), $\beta = 0.8$ (—·), $\beta = 0.7$ (···), $\beta = 0.3$ upper solid line.

a bivariate Gaussian. Figure 10 shows the fundamental diagram for fixed desired velocity distribution ($H(w)$ is a Gaussian with $\bar{w} = 35$ m/s and $H(w) \equiv 0$ for $w \notin [20 \text{ m/s}, 50 \text{ m/s}]$), but different values of the parameter β. The case $\beta = 1.0$, i.e., "perfect braking," is shown for reference. Wegener and Klar[22] chose $\beta = 0.3$ in their model. Although this seems to us to be very low, we also investigate this case to keep in contact with their work.

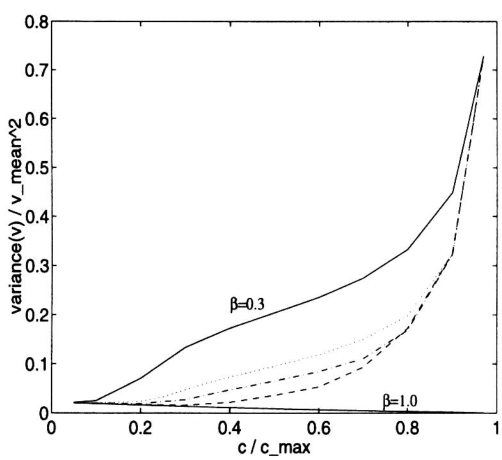


Fig. 13. $\Theta_{vv}/\bar{v}^2$ as a function of density: $\beta = 1.0$ lower solid line, $\beta = 0.9$ (— —), $\beta = 0.8$ (—·), $\beta = 0.7$ (···), $\beta = 0.3$ upper solid line.

As main result we find that now $\bar{v} \to 0$ for $c \to \hat{c}$, although the lowest desired speed $w_0 \gg 0$. The fundamental diagram declines faster for smaller values of β. Figure 11 depicts the flux $q = c\bar{v}$ as a function of the density. If we take a look at the variance as a function of density (Fig. 12), we find a plateau for $\beta = 0.8$ and $\beta = 0.9$, whereas for $\beta = 0.7$ and $\beta = 0.3$ the variance is even increasing. This reflect the fact that vehicles can now be scattered to velocities smaller than the minimal desired velocity, thus broadening the velocity distribution. In Fig. 13 we show the variance relative to the squared mean velocity as a function of density and in Fig. 14 we show the velocity distribution $F(v)$ at $c = 0.5\hat{c}$ for different values of β. For $\beta = 0.3$ the velocity distribution shows a significant skewness, which seems not be supported by data.[23]

Now we go on to investigate the influence of the shape of the desired velocity distribution on the solution. We choose $\beta = 0.9$ and a uniform desired velocity distribution with the same mean value and variance as the Gaussian in the former case. The surprising result is that the exact shape of the desired velocity distribution has not much effect on the mean-velocity–density relation (Fig. 15) and on the variance as a function of the density (Fig. 16). The covariances as functions of the density are almost identical in both cases (Fig. 16). Furthermore, we find that the velocity distribution $F(v)$ is Gaussian shaped for medium and high densities, i.e., the Gaussian shape of the equilibrium velocity distribution is not just a consequence of the imposed desired velocity distribution, but an inherent property of the improved kinetic equation. This is certainly a consequence

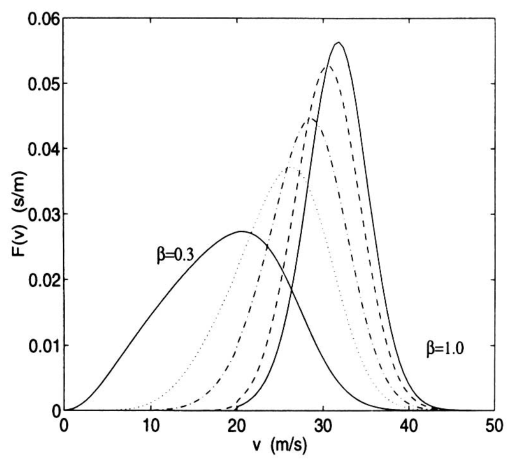


Fig. 14. Velocity distribution $F(v)$ at $c = 0.5\hat{c}$: $\beta = 1.0$ right solid line, $\beta = 0.9$ (— —), $\beta = 0.8$ (—·), $\beta = 0.7$ (···), $\beta = 0.3$ left solid line.

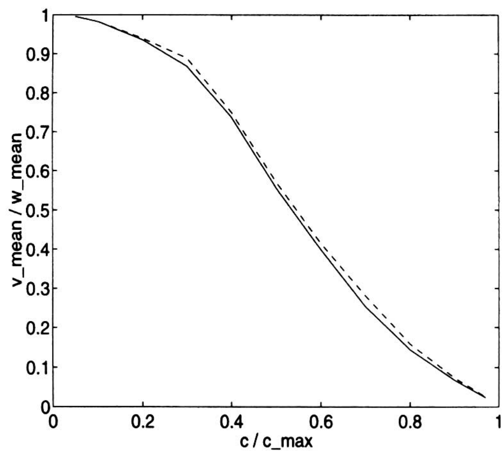


Fig. 15. Mean-velocity-density relation with $\bar{w} = 35$ m/s and $\Theta_{ww} = 25.8$ m^2/s^2: Gaussian desired velocity distribution (–), uniform desired velocity distribution (– –).

of the randomness introduced through the vehicular chaos assumption and the imperfect velocity adaptation, but it leads to velocity distributions known from traffic measurements. In CA-models and car-following models one finds in the stationary state a bimodal velocity distribution since the vehicles tend to accumulate at the highest and the lowest possible velocity

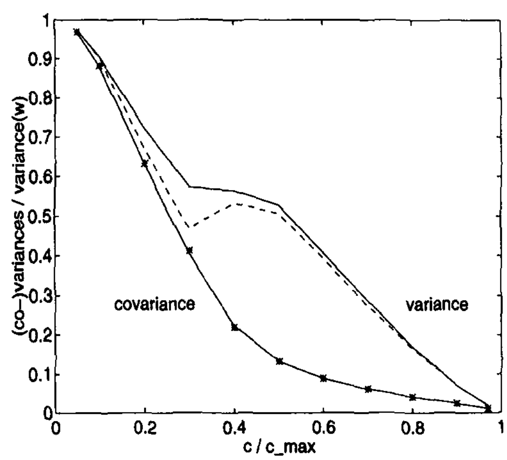


Fig. 16. Velocity-variance as a function of density with $\bar{w} = 35$ m/s and $\Theta_{ww} = 25.8$ m^2/s^2: Gaussian desired velocity distribution (–), uniform desired velocity distribution (– –). Covariance as a function of density: Gaussian desired velocity distribution (–), uniform desired velocity distribution (*).

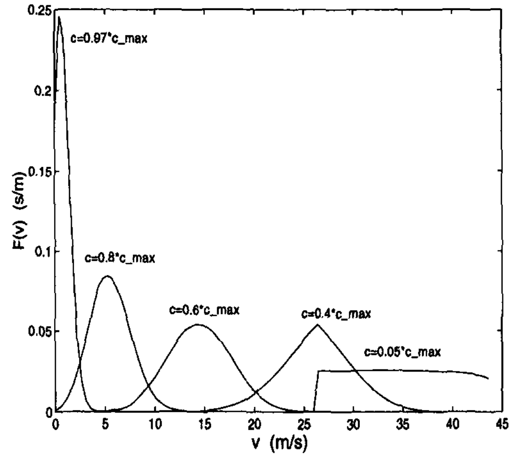


Fig. 17. Velocity distribution $F(v)$ at different densities. $\bar{w} = 35$ m/s, $\Theta_{ww} = 25.8\ \mathrm{m^2/s^2}$ (uniform desired velocity distribution).

(stop-and-go waves). The reason for this is the use of single lane models with periodic boundary conditions. Only recently, multilane and multi-species microscopic models have been suggested,[24, 25] but so far the velocity distributions for open boundary conditions and different driving behavior have not been investigated. Having found an interaction model which yields realistic results in the stationary homogeneous case over the whole density regime, we now turn to the macroscopic model.

V. MACROSCOPIC MODEL

The whole improved kinetic equation now reads

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) g(x, v, w, t) + \frac{\partial}{\partial v}\left(\frac{w-v}{T}\, g(x, v, w, t)\right)$$
$$= \sigma \int_v^{v/\beta} dv_1\, f(x, v_1, t) \int_{v_1}^{\infty} dv_2 (v_2 - v_1)\, g(x, v_2, w, t) \frac{1}{v_1(1-\beta)}$$
$$- \sigma g(x, v, w, t) \int_0^v dv_1 (v - v_1)\, f(x, v_1, t) \tag{5.1}$$

As in ref. 10 we derive now a macroscopic model by taking the moments of Eq. (5.1). For brevity the time and space arguments of the distribution functions are omitted. An integration over $\int dv \int dw\, v^k w^l$ of the left-hand convective part of Eq. (5.1) leads after integration by parts and vanishing surface terms to (see also refs. 10 and 11)

$$\int dv \int dw\, v^k w^l [LHS(5.1)]$$

$$= \frac{\partial}{\partial t} m_{k,l} + \frac{\partial}{\partial x} m_{k+1,l} + \frac{k}{T}(m_{k,l} - m_{k-1,l+1}) \tag{5.2}$$

$m_{k,l} = \int dv \int dw\, v^k w^l g$ denotes the k, lth moment, with $m_{-1,1} := 0$. The integration over the right-hand side of Eq. (5.1), i.e., the collision operator, yields

$$\int dv \int dw\, v^k w^l [RHS(5.1)]$$

$$= \sigma \int_0^\infty dv \int_0^\infty dw\, v^k w^l \int_v^{v/\beta} dv_1\, f(v_1) \int_{v_1}^\infty dv_2 (v_2 - v_1) \frac{g(v_2, w)}{v_1(1-\beta)}$$

$$- \sigma \int_0^\infty dv \int_0^\infty dw\, v^k w^l g(v, w) \int_0^v dv_1 (v - v1)\, f(v_1)$$

$$= \sigma \mu_k \int_0^\infty dv \int_0^\infty dw\, v^k f(v) \int_v^\infty dv_1 (v_1 - v)\, w^l g(v_1, w)$$

$$- \sigma \int_0^\infty dv \int_0^\infty dw\, v^k w^l g(v, w) \int_0^v dv_1 (v - v_1)\, f(v_1) \tag{5.3}$$

with

$$\mu_k = \frac{1}{k+1} \sum_{i=0}^{k} \beta^i, \qquad \mu_k < \mu_{k+1}, \quad k \in \mathbb{N}$$

In order to evaluate integrals in Eq. (5.3) we make use of a cumulant expansion of the distribution functions g and f. We then neglect third and higher order cumulants. This is equivalent to a Gaussian approximation of the distribution functions and is justified by the result of Section III. For the velocity equation, i.e., the case $k = 1$ and $l = 0$, the derivation is briefly demonstrated in the Appendix (see also ref. 10 for a more detailed description). The higher order cumulant equations are obtained similarly and we eventually find the macroscopic equations

$$\frac{\partial}{\partial t}c+\frac{\partial}{\partial x}(c\bar{v})=0 \tag{5.4}$$

$$\begin{aligned}\partial_t\bar{v}+\bar{v}\,\partial_x\bar{v}=&\frac{\bar{w}-\bar{v}}{T}-\frac{1}{c}\partial_x(c\Theta_{vv})\\&-\sigma c\left\{\frac{\mu_1+1}{2}\Theta_{vv}+\frac{1-\mu_1}{\sqrt{\pi}}\bar{v}\sqrt{\Theta_{vv}}\right\}\end{aligned} \tag{5.5}$$

$$\partial_t\bar{w}+\bar{v}\,\partial_x\bar{w}=-\frac{1}{c}\partial_x(c\Theta_{vw}) \tag{5.6}$$

$$\begin{aligned}\partial_t\Theta_{vv}+\bar{v}\,\partial_x\Theta_{vv}=&\frac{2}{T}(\Theta_{vw}-\Theta_{vv})-2\Theta_{vv}\partial_x\bar{v}-\sigma c\left\{(\mu_2-\mu_1)\,\bar{v}\Theta_{vv}\right.\\&\left.+\frac{3(1-\mu_2)}{2\sqrt{\pi}}\sqrt{\Theta_{vv}^3}+\frac{2\mu_1-\mu_2-1}{\sqrt{\pi}}\bar{v}^2\sqrt{\Theta_{vv}}\right\}\end{aligned} \tag{5.7}$$

$$\partial_t\Theta_{ww}+\bar{v}\,\partial_x\Theta_{ww}=-2\Theta_{vw}\,\partial_x\bar{w} \tag{5.8}$$

$$\begin{aligned}\partial_t\Theta_{vw}+\bar{v}\,\partial_x\Theta_{vw}=&-\Theta_{vw}\,\partial_x\bar{v}-\Theta_{vv}\,\partial_x\bar{w}-\frac{1}{T}(\Theta_{vw}-\Theta_{ww})\\&-\frac{\sigma c}{2}\left\{(1-\mu_1)\,\bar{v}\Theta_{vw}+\frac{\mu_1+3}{\sqrt{\pi}}\Theta_{vw}\sqrt{\Theta_{vv}}\right\}\end{aligned} \tag{5.9}$$

For $\beta=0$, i.e., "perfect braking," follows that $\mu_k=1$, $\forall k$, and Eqs. (5.4)–(5.9) reduce to the basic model derive in ref. 10.

A. The Time-Independent Homogeneous Equations

In the stationary homogeneous case Eqs. (5.4)–(5.9) reduce to

$$\bar{w}-\bar{v}=\sigma cT\left\{\frac{\mu_1+1}{2}\Theta_{vv}+\frac{1-\mu_1}{\sqrt{\pi}}\bar{v}\sqrt{\Theta_{vv}}\right\} \tag{5.10}$$

$$\begin{aligned}\Theta_{vw}-\Theta_{vv}=&\frac{T\sigma c}{2}\left\{(\mu_2-\mu_1)\,\bar{v}\Theta_{vv}+\left(\frac{3}{2}\right)\frac{1-\mu_2}{\sqrt{\pi}}\sqrt{\Theta_{vv}^3}\right.\\&\left.+\frac{2\mu_1-\mu_2-1}{\sqrt{\pi}}\bar{v}^2\sqrt{\Theta_{vv}}\right\}\end{aligned} \tag{5.11}$$

$$\Theta_{ww}-\Theta_{vw}=\frac{\sigma cT}{2}\left\{(1-\mu_1)\,\bar{v}\Theta_{vw}+\frac{\mu_1+3}{\sqrt{\pi}}\Theta_{vw}\sqrt{\Theta_{vv}}\right\} \tag{5.12}$$

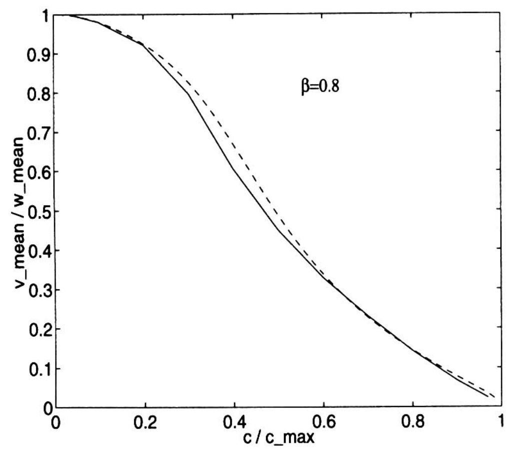


Fig. 18. Mean-velocity-density relation: Kinetic equilibrium solution (solid line), macroscopic equilibrium solution (dashed line).

First, let us investigate the behavior near the maximal density $\hat{c}$. For $c \to \hat{c}$, we obtain from Eq. (5.12) $\bar{v}\Theta_{vw} \sim \mathcal{O}(\chi^{-1})$ and $\Theta_{vw}\sqrt{\Theta_{vv}} \sim \mathcal{O}(\chi^{-1})$. Equation (5.10) yields $\Theta_{vv} \sim \mathcal{O}(\chi^{-1})$ and $\bar{v}\sqrt{\Theta_{vv}} \sim \mathcal{O}(\chi^{-1})$, thus $\bar{v} \sim \mathcal{O}(\chi^{-1/2})$ and $\Theta_{vw} \sim \mathcal{O}(\chi^{-1/2})$. This means that the mean velocity and the variance both go to zero when the density approaches the maximal density. Unfortunately,

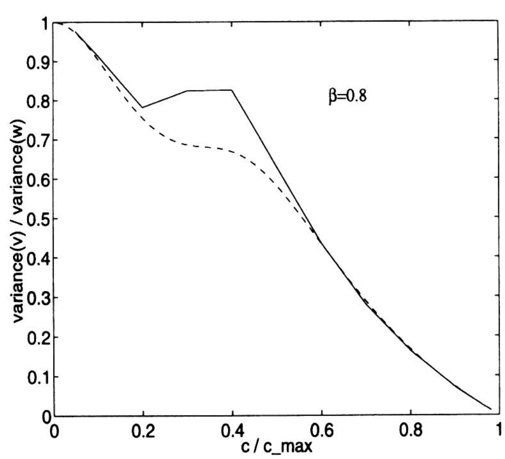


Fig. 19. Variance-density relation: Kinetic equilibrium solution (solid line), macroscopic equilibrium solution (dashed line).

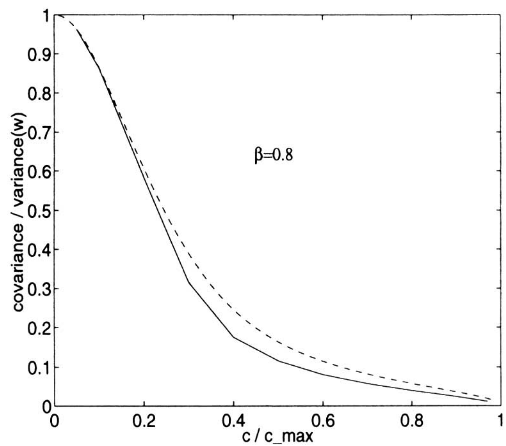


Fig. 20. Covariance-density relation: Kinetic equilibrium solution (solid line), macroscopic equilibrium solution (dashed line).

Eqs. (5.10)–(5.12) can only be solve numerically for given values of c, $\bar{w}$, Θ_{ww} and β. When we compare the solutions of the macroscopic equilibrium Eqs. (5.10)–(5.12) with averaged values obtained from the kinetic equilibrium equation [LHS(3.4) = RHS(4.1)], we find a good agreement of the mean velocity-density relation (Fig. 18). The variances (Figs. 19 and 20) differ a bit more, but apparently this has not much impact on the $\bar{v}-c$-relation. We conclude that a bivariate Gaussian distribution is a good approximation of the local equilibrium solution of the kinetic traffic Eq. (5.1).

VI. CONCLUSIONS

In the derivation of macroscopic traffic models from a kinetic equation the stationary homogeneous case plays a crucial role as closure assumption when truncating the hierarchy of moment equations. Results from traffic measurements suggest a normal velocity distribution, but the stationary homogeneous problem itself has not been thouroughly investigated so far. The present paper serves to close this gap. We solved numerically Paveri–Fontana's kinetic traffic equation for the equilibrium case and find that the solution shows a separation of free and interacting vehicles. Since we work with a quasi-one-dimensional kinetic equation this separation holds locally on a coarse grained scale. We then introduce modifications to the kinetic equation to remedy the main deficiencies of Paveri-Fontana's original version. We find that the equilibrium solution of the improved kinetic equation can be approximated very well by a bivariate Gaussian distribution. Furthermore, the velocity distribution turns out to have a Gaussian shape, which is not just a consequence of the imposed desired velocity distribution, but an inherent property of the improved kinetic equation. Thus the equilibrium velocity distribution is in agreement with empirical data. Nevertheless, the vehicular chaos assumption remains somewhat unsatisfactory, since in vehicular traffic correlations between interacting vehicles should play a more important role than in the classical interaction process. So, two-vehicle distribution functions and correlations remain important issues to be investigated within traffic theory. From the improved kinetic equation we derive a macroscopic model. The macroscopic equilibrium solution shows very good agreement with the calculated cumulants of the kinetic equilibrium solution, which justifies the closure assumption. We thus have found a macroscopic model based on a microscopic considerations which yields realistic results over the whole density regime. A comparison of the $\bar{v}-\rho$ relation of Fig. 18 with mean-velocity-density relations extracted from empirical data show that the latter decline much faster. It is now rather tempting to remedy this defect by choosing a small value for the parameter β (see Fig. 10), but this has then to be interpreted as a very strong over-reaction of the drivers. More appropriate seems to be a variation of the cross section σ (see Section IV A). A modified passing probability $P(c) = 1-(c/\hat{c})^{\alpha}$ with $0<\alpha<1$, or a mean velocity dependent correction factor χ due to a safety distance (see ref. 10) would also lead to faster decreasing $\bar{v}-\rho$ relation. In Section IV B we only considered the stationary homogeneous effect of a finite vehicle length in form of a modified cross section. In the dynamical case we also have to consider the different position of the collision partners. This can be approximated by a Taylor expansion in space of the distribution function and has been carried out in ref. 10 for the case of perfect velocity adaptation. Imperfect velocity adaptation can be treated similarly and only yields higher order corrections to the the Taylor expansion, so for the improved macroscopic model we can use the additional gradient terms derived in ref. 10 to consider the dynamical effect of a finite vehicle length.

APPENDIX: DERIVATION OF THE RHS OF THE VELOCITY EQUATION

For $k=1$ and $l=0$, the first term of Eq. (5.3) can be written as (we omit the factors σ, μ_1, and time and position dependence):

$$
\begin{aligned}
[\text{1st term}] &= \int_0^\infty dv \int_0^\infty dw\, v f(v) \int_v^\infty dv_1 (v_1 - v)\, g(v_1, w) \\
&= \int_0^\infty dv\, f(v) \int_0^v dv_1 (v v_1 - v_1^2)\, f(v_1) \\
&= \int_{-\infty}^\infty \frac{dz}{2\pi i} \frac{1}{z - i\varepsilon} \int_0^\infty dv\, f(v) \int_0^\infty dv_1 (v v_1 - v_1^2)\, e^{iz(v - v_1)} f(v_1) \\
&= \int_{-\infty}^\infty \frac{dz}{2\pi i} \frac{1}{z - i\varepsilon} \left\{ \left[\frac{\partial}{\partial(iz)} \int dv\, e^{izv} f(v) \right] \left[\frac{\partial}{\partial(-iz)} \int dv_1\, e^{-izv_1} f(v_1) \right] \right. \\
&\quad \left. - \int dv\, e^{izv} f(v) \left[\frac{\partial^2}{\partial(-iz)^2} \int dv_1\, e^{-izv_1} f(v_1) \right] \right\} \\
&\approx c^2 \int_{-\infty}^\infty \frac{dz}{2\pi i} (-\Theta_{vv} + 2i\bar{v}\Theta_{vv} z + 2\Theta_{vv}^2 z^2) \frac{\exp(-z^2 \Theta_{vv})}{z - i\varepsilon} \qquad \text{(A1)}
\end{aligned}
$$

From the first line to the second line we have used the Fourier transform of the step-function. From the third to the fourth line, a cumulant expansion of $f(v_1)$ and $f(v)$ was used omitting third and higher oder cumulants.

The second term is treated similarly and we find

$$
\begin{aligned}
[\text{2nd term}] &= \int_0^\infty dv \int_0^\infty dw\, v g(v, w) \int_0^v dv_1 (v - v_1)\, f(v_1) \\
&\approx c^2 \int_{-\infty}^\infty \frac{dz}{2\pi i} (-\Theta_{vv} - 2i\bar{v}\Theta_{vv} z + 2\Theta_{vv}^2 z^2) \frac{\exp(-z^2 \Theta_{vv})}{z - i\varepsilon} \qquad \text{(A2)}
\end{aligned}
$$

The remaining integrals can easily be evaluated by using methods of complex function theory. This yields

$$
[RHS(5.3)] \approx -\sigma c^2 \left(\frac{\mu_1 + 1}{2} \Theta_{vv} + \frac{\mu_1 - 1}{\sqrt{\pi}} \bar{v} \sqrt{\Theta_{vv}} \right) \qquad \text{(A3)}
$$

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